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Solvable potentials of shape invariance in two steps

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Abstract
Within the framework of supersymmetric quantum mechanics, we obtain a class of solvable potentials of shape invariance in two steps, where the parameters a_1 and a_2 of partner potentials are related to each other by translation $a_2 = a_1 + \alpha$. It is found that discontinuity at some x-points is a characteristic of the two-step superpotentials, therefore giving rise to Dirac delta-function singularities to the corresponding potentials.

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1. Introduction

The number of solvable potentials for the Schrödinger equation in nonrelativistic quantum mechanics is rather limited. In this regard, the ideal of shape invariance condition [1] in supersymmetric quantum mechanics [2, 3] becomes very useful, because it leads immediately to an integrability condition. Supersymmetric quantum mechanics (SUSYQM) was initially proposed to study dynamical supersymmetry (SUSY) breaking in quantum field theory [2]. It was soon realized that SUSYQM by itself was very interesting. For instance, the formalism of SUSYQM enables us to construct a family of isospectral Hamiltonians starting from a given one-dimensional Hamiltonian [4]. For a review of SUSYQM, please refer to [5–7] and references therein. Later on, the concept of shape invariance within the structure of SUSYQM was introduced by Gendenshtein [1]. It is readily shown that for any shape invariant potential in SUSYQM, the energy eigenvalues and eigenfunctions can be obtained algebraically if SUSY remains unbroken. Shape invariance condition is also studied in the framework of the so-called fractional SUSYQM of order k ($k = 3, 4, \ldots$) [8]. In order to pass from ordinary SUSYQM to fractional SUSYQM of order k, one has to replace Z_2-grading of the relevant Hilbert space by a Z_k-grading [9, 10].

In the first two sections (sections 1 and 2) of the present paper, we shall review and reproduce some results of [11]. The key ingredient in solving the exact eigenvalue problems in SUSYQM is the connection between the ground-state wavefunction and the potential (up to a constant). So, let us consider a Hamiltonian, in units of $\hbar = 2m = 1$, having the following factorizable form:

$$H_- = -\frac{d^2}{dx^2} + V_-(x) \equiv A^\dagger A, \tag{1}$$

where the operators A and $A^\dagger$ are defined by

$$A = \frac{d}{dx} + W(x), \qquad A^\dagger = -\frac{d}{dx} + W(x). \tag{2}$$

Hence the potential $V_-(x)$ is given as

$$V_-(x) = W^2(x) - W'(x), \tag{3}$$

where $W'(x) \equiv \frac{d}{dx}W(x)$. The quantity $W(x)$ is generally referred to as the superpotential in SUSYQM literature. In the case of unbroken SUSY, we note that the unnormalized wavefunction, constructed out of the superpotential $W(x)$,

$$\psi_0^{(-)}(x) \propto \exp\left[-\int^x W(y)\,dy\right] \tag{4}$$

is the nodeless zero-energy ground-state eigenfunction for the Hamiltonian H_-, since the equation $A\psi_0^{(-)}(x) = 0$ is fulfilled.

To establish a SUSY theory out of the original Hamiltonian H_-, we define another Hamiltonian by simply reversing the order of A and $A^\dagger$. It is

$$H_+ \equiv AA^\dagger = -\frac{d^2}{dx^2} + V_+(x). \tag{5}$$

A little algebra shows that

$$V_+(x) = W^2(x) + W'(x). \tag{6}$$

At this time, the zero-energy ground state for H_+ is the unnormalized wavefunction $\psi_0^{(+)}(x) \propto \exp\left[+\int^x W(y)\,dy\right]$, because of $A^\dagger\psi_0^{(+)}(x) = 0$.

The pair of potentials $V_-(x)$ and $V_+(x)$ is named as SUSY partner potentials. The Hamiltonians H_- and H_+ are thus called SUSY partner Hamiltonians. One special feature about the SUSY theory is that except for an additional zero-energy eigenstate of H_-, the pair of partner Hamiltonians is found to have exactly the same energy eigenvalues. Explicitly, the eigenstates of H_- and H_+ are related to each other by ($n = 1, 2, 3, \ldots$)

$$\psi_{n-1}^{(+)}(x) = \left[E_n^{(-)}\right]^{-1/2} A\psi_n^{(-)}(x), \qquad \psi_{n+1}^{(-)}(x) = \left[E_n^{(-)}\right]^{-1/2} A^\dagger\psi_n^{(+)}(x). \tag{7}$$

Here, the eigenvalues are $E_0^{(-)} = 0$ and $E_{n-1}^{(+)} = E_n^{(-)}$. As a result, if we know all the eigenfunctions of H_- we can determine all the eigenfunctions of H_+, and vice versa, except for the zero-energy ground-state eigenfunction $\psi_0^{(-)}(x)$ of H_-.

An integrability condition called shape invariance can be imposed to further relate the pair of SUSY partner potentials (3) and (6). Using this condition, one can easily determine the entire spectrum of H_- algebraically [1, 12]. Though the general problem of shape invariance condition is yet to be solved, the partial lists of classification of such solvable potentials have been constructed. We shall discuss the shape invariance condition in a bit detail in section 2.

In the present paper, we study shape invariance condition in two steps and obtain a class of solvable potentials, in which the parameters a_1 and a_2 of partner potentials are related to each other by translation $a_2 = a_1 + \alpha$. It is found that some SUSY preserving singular potentials, such as singular harmonic oscillator, singular Pöschl–Teller I and singular Pöschl–Teller II potentials discussed in [13], belong to this class as a special case. One important aspect regarding the translation class of shape invariance condition in two steps is that discontinuity at some x-points is characteristic of the shape invariant superpotentials, and thus leads to Dirac delta-function singularities to the corresponding potentials. The organization of the paper is as follows. In section 2, we review the formalism of shape invariance condition, to which the partner potentials are related. In section 3, using an ansatz, we obtain a class of SUSY preserving potentials by solving the shape invariance condition in two steps. Section 4 presents the conclusions.

2. Formalism of shape invariance condition

The concept of shape invariance within the formulation of SUSYQM was first introduced by Gendenshtein [1]. Generally speaking, the pair of partner potentials $V_\pm(x)$ defined in equations (3) and (6) are said to be shape invariant, if both of them are similar in shape but differ only up to a change of parameters and additive constants. For instance, $V_-(x)$ and $V_+(x)$ are said to be shape invariant, if they satisfy the relation

$$V_+(x, a_1) = V_-(x, a_2) + R(a_1), \tag{8}$$

where a_1 is a set of parameters, $a_2 = f(a_1)$ is a function of a_1 and the remainder $R(a_1)$ is independent of x. We call equation (8) the one-step shape invariance condition, since the partner potentials $V_-(x, a_2)$ and $V_+(x, a_1)$ are related to each other by only one relation.

By using the shape invariance condition (8), the entire spectrum of the Hamiltonian H_- can be obtained algebraically. It is found that the complete energy eigenvalues of H_- are, for $n = 1, 2, 3, \ldots,$

$$E_0^{(-)} = 0, \qquad E_n^{(-)} = \sum_{k=1}^{n} R(a_k), \tag{9}$$

and the corresponding nth unnormalized energy eigenfunction is given by

$$\psi_n^{(-)}(x, a_1) \propto A^\dagger(a_1)A^\dagger(a_2)\cdots A^\dagger(a_n)\psi_0^{(-)}(x, a_{n+1}). \tag{10}$$

Here, we have suppressed the x-dependence in the operators $A^\dagger(x, a_i)$ to simplify the notation. We recall that the zero-energy ground-state eigenfunction $\psi_0^{(-)}(x, a_1)$ is expressible in terms of the superpotential in equation (4).

The classification of various solutions to the shape invariance condition in one-step (8) has been established. Four classes of solvable shape invariant potentials that retain SUSY are found and discussed. The first class [14], where the parameters a_1 and a_2 are related to each other by translation ($a_2 = a_1 + \alpha$), contains all the analytically solvable potentials known in the context of nonrelativistic quantum mechanics. In the second class [11, 15], the parameters a_1 and a_2 are related by the scaling ($a_2 = qa_1$, $0 < q < 1$). In the third class [11], the parameters a_1 and a_2 are related by the possibilities: $a_2 = qa_1^p$, for $0 < q < 1$ and $p = 2, 3, \ldots$ and $a_2 = qa_1/(1 + pa_1)$. It is found that classes two and three contain those potentials that are not obtainable in terms of elementary functions but only in a series form. In regard to the last class [16], the parameter repeats itself after a cycle of p iterations, one thus has the relations: $a_1 = a_{p+1}$ and $f(a_1) = a_2 = a_{p+2}$.[1]

We can in principle go one step further to obtain more solvable shape invariant potentials by simply extending the concept of shape invariance condition to two and even multi-steps. The procedure of generalization is rather straightforward. Take the shape invariance condition in two steps as an example. In the situation of unbroken SUSY, we consider two superpotentials

[1] Strictly speaking, these four classes can be transformed to one another by suitable reparameterizations. For example, the scaling form $a_2 = qa_1$ can be rearranged into the translation form $a_2' = a_1' + \alpha$ by taking logarithms.

$W(x,a_1)$ and $\tilde{W}(x,a_1)$ such that the derived potentials $V_+(x,a_1)$ and $\tilde{V}_-(x,a_1)$ are the same up to an additive constant

$$V_+(x,a_1) = \tilde{V}_-(x,a_1) + R(a_1), \tag{11}$$

or, equivalently, in terms of the superpotentials $W(x,a_1)$ and $\tilde{W}(x,a_1)$

$$W^2(x,a_1) + W'(x,a_1) = \tilde{W}^2(x,a_1) - \tilde{W}'(x,a_1) + R(a_1). \tag{12}$$

The shape invariance condition then imposes this relation

$$\tilde{V}_+(x,a_1) = V_-(x,a_2) + \tilde{R}(a_1), \tag{13}$$

that is, alternatively

$$\tilde{W}^2(x,a_1) + \tilde{W}'(x,a_1) = W^2(x,a_2) - W'(x,a_2) + \tilde{R}(a_1). \tag{14}$$

Equations (11) and (13) (or (12) and (14)) together are called the shape invariance condition in two steps, because the partner potentials $V_-(x,a_2)$ and $V_+(x,a_1)$ now are related to each other by two relations, not just by one.

Similar to shape invariance in one step, the energy eigenvalues and eigenfunctions of the potential $V_-(x,a_1)$ for the shape invariance condition in two steps can also be obtained algebraically, when equations (11) and (13) simultaneously hold. It can be shown that on solving the two equations the eigenvalues are $(n = 0, 1, 2, \ldots)$

$$E_{2n}^{(-)} = \sum_{k=1}^{n}[R(a_k) + \tilde{R}(a_k)], \tag{15}$$

$$E_{2n+1}^{(-)} = \sum_{k=1}^{n}[R(a_k) + \tilde{R}(a_k)] + R(a_{n+1}), \tag{16}$$

and the corresponding unnormalized eigenfunctions are

$$\psi_{2n}^{(-)}(x,a_1) \propto [A^\dagger(a_1)\tilde{A}^\dagger(a_1)]\cdots[A^\dagger(a_n)\tilde{A}^\dagger(a_n)]\psi_0^{(-)}(x,a_{n+1}), \tag{17}$$

$$\psi_{2n+1}^{(-)}(x,a_1) \propto [A^\dagger(a_1)\tilde{A}^\dagger(a_1)]\cdots[A^\dagger(a_n)\tilde{A}^\dagger(a_n)]A^\dagger(a_{n+1})\tilde{\psi}_0^{(-)}(x,a_{n+1}), \tag{18}$$

where to avoid notational complexity, the x-dependence of the operators $A^\dagger(x,a_i)$ and $\tilde{A}^\dagger(x,a_i)$ is suppressed. $\psi_0^{(-)}(x,a_1)$ and $\tilde{\psi}_0^{(-)}(x,a_1)$, expressible in terms of the corresponding superpotentials by using equation (4), denote the zero-energy ground-state eigenfunctions for the potentials $V_-(x,a_1)$ and $\tilde{V}_-(x,a_1)$, respectively. Here, we note the different structures of energy eigenvalues, equations (15) and (16), between the even number eigenstates $\psi_{2n}^{(-)}(x,a_1)$ and the odd number ones $\psi_{2n+1}^{(-)}(x,a_1)$.

As far as the classification of various solutions of shape invariance condition in two steps is concerned, two classes of solutions have been obtained up-to-date [11]. They are the solutions where the parameters a_1 and a_2 are related to each other by scaling $a_2 = qa_1$ and where those by the relation $a_2 = qa_1^p$, for p a positive integer. To author's knowledge, no solution other than these two classes has ever been constructed. For example, no solution has been reported in the translation class for the shape invariance condition in two and even multi-steps. Hence, in section 3 we shall investigate this particular problem and construct explicitly the solvable potentials of shape invariance in two steps, where the parameters a_1 and a_2 are related by translation ($a_2 = a_1 + \alpha$).

3. Solutions of shape invariance in two steps

This section is aimed to find the solvable solutions to the shape invariance condition in two steps, for the class that the parameters a_1 and a_2 are related by translation. Mathematically, the condition is described by equations (11) and (13) (or (12) and (14)). It turns out that some shape invariant potentials in two steps can be constructed, in which the singular potentials discovered recently are found to be included as a special case.

To find the two-step shape invariant potentials for the translation class ($a_2 = a_1 + \alpha$, where a_1 is an arbitrary parameter and α is a constant), let us make an ansatz. We assume that the superpotentials $W(x, a_1)$ and $\tilde{W}(x, a_1)$ admit the following expansions in the a_1 parameter[2]:

$$W(x, a_1) = g_0(x) + g_1(x)a_1, \qquad \tilde{W}(x, a_1) = \tilde{g}_0(x) + \tilde{g}_1(x)a_1, \tag{19}$$

where the functions $g_i(x)$ and $\tilde{g}_i(x)$ for $i = 0, 1$ are to be determined. Similarly, the remainders $R(a_1)$ and $\tilde{R}(a_1)$ are assumed to have the expansions

$$R(a_1) = R_0 + R_1 a_1 + R_2 a_1^2, \qquad \tilde{R}(a_1) = \tilde{R}_0 + \tilde{R}_1 a_1 + \tilde{R}_2 a_1^2, \tag{20}$$

where the coefficients R_i and $\tilde{R}_i$ for $i = 0, 1, 2$ are constants, independent of the parameter a_1. Note that in this ansatz the superpotentials and the remainders are treated differently in the expansions in powers of a_1.

By applying the above ansatz for the superpotentials $W(x, a_1)$ and $\tilde{W}(x, a_1)$ and for the remainders $R(a_1)$ and $\tilde{R}(a_1)$ to the shape invariance condition in two steps, we should find some equations satisfied by the unknown functions $g_i(x)$ and $\tilde{g}_i(x)$. To be more precise, when substituting equations (19) and (20) into equations (12) and (14) and equating the coefficients of the powers of a_1, we obtain five equations as follows. At the $(a_1)^2$ order, we find one equation

$$g_1^2 - \tilde{g}_1^2 = R_2 = -\tilde{R}_2. \tag{21}$$

At the $(a_1)^1$ order, we have two equations

$$(g_1' + \tilde{g}_1') + 2(g_0 g_1 - \tilde{g}_0 \tilde{g}_1) = R_1, \tag{22}$$

$$(g_1' + \tilde{g}_1') - 2(g_0 g_1 - \tilde{g}_0 \tilde{g}_1) - 2\alpha g_1^2 = \tilde{R}_1. \tag{23}$$

At the $(a_1)^0$ order, we get

$$(g_0' + \tilde{g}_0') + \left(g_0^2 - \tilde{g}_0^2\right) = R_0, \tag{24}$$

$$(g_0' + \tilde{g}_0') - \left(g_0^2 - \tilde{g}_0^2\right) + \alpha\left(g_1' - 2g_0 g_1 - \alpha g_1^2\right) = \tilde{R}_0. \tag{25}$$

The solutions of the shape invariant potentials in two steps can be readily obtained by simultaneously solving the set of above equations, equations (21)–(25). Let us solve these equations one by one. First, from equation (21) we obtain

$$\tilde{g}_1(x) = \pm\sqrt{g_1^2(x) - R_2}. \tag{26}$$

The $\tilde{g}_1(x)$ function is expressed in terms of $g_1(x)$ and can have either the same or the opposite sign to the $g_1(x)$ function.

[2] The ansatz is motivated by its counterpart in the translation class of the one-step shape invariance condition. For instance, if we take in equation (8) that $W(x, a_1) = g_0(x) + g_1(x)a_1$ and $R(a_1) = R_0 + R_1 a_1$, then many known solvable solutions can be easily deduced by different choices of α, R_0 and R_1. It is noted that another ansatzes might lead to more solvable shape invariant potentials in two steps, but at present the existence of such ansatzes is not known to the author.

Secondly, the addition of equations (22) and (23) yields a first-order differential equation satisfied by $g_1(x)$ as

$$\left[1 \pm \frac{g_1(x)}{\sqrt{g_1^2(x) - R_2}}\right] g_1'(x) = \frac{1}{2}(R_1 + \tilde{R}_1) + \alpha g_1^2(x), \tag{27}$$

or, equivalently, that by $\tilde{g}_1(x)$ as

$$\left[1 \pm \frac{\tilde{g}_1(x)}{\sqrt{\tilde{g}_1^2(x) - \tilde{R}_2}}\right] \tilde{g}_1'(x) = \frac{1}{2}(R_1 + \tilde{R}_1 + 2\alpha R_2) + \alpha \tilde{g}_1^2(x), \tag{28}$$

where we have used equation (26), in which the $\pm$ sign on the left-hand side of the above both equations is defined. The SUSY preserving solutions of $g_1(x)$ and $\tilde{g}_1(x)$ can be easily obtained by solving the two equations. We shall come back to this point later.

Thirdly, the remaining three equations are reshuffled into

$$4(g_0 g_1 - \tilde{g}_0 \tilde{g}_1) + 2\alpha g_1^2 = R_1 - \tilde{R}_1, \tag{29}$$

$$2\left(g_0^2 - \tilde{g}_0^2\right) - \alpha\left(g_1' - 2g_0 g_1 - \alpha g_1^2\right) = R_0 - \tilde{R}_0, \tag{30}$$

$$2(g_0 + \tilde{g}_0)' + \alpha\left(g_1' - 2g_0 g_1 - \alpha g_1^2\right) = R_0 + \tilde{R}_0. \tag{31}$$

Therefore, via the first two equations (29) and (30), we can in principle express the unknown functions $g_0(x)$ and $\tilde{g}_0(x)$ in terms of $g_1(x)$, $\tilde{g}_1(x)$, and the coefficients R_i and $\tilde{R}_i$ $(i = 0, 1)$. These results can then be substituted into the third equation (31), which serves as a consistency condition, to obtain the explicit expressions for $g_0(x)$ and $\tilde{g}_0(x)$. The detailed computation is rather straightforward, but a bit tedious. After the calculation, which is omitted here, we get $g_0(x)$ and $\tilde{g}_0(x)$ of the following forms:

$$g_0(x) = \frac{1}{4R_2}[(R_1 - \tilde{R}_1 - 2\alpha R_2) g_1(x) \pm \mathcal{F}(x) \tilde{g}_1(x)], \tag{32}$$

$$\tilde{g}_0(x) = \frac{1}{4R_2}[(R_1 - \tilde{R}_1) \tilde{g}_1(x) \pm \mathcal{F}(x) g_1(x)], \tag{33}$$

where the function $\mathcal{F}(x)$ is defined by

$$\mathcal{F}(x) \equiv (R_1 + \tilde{R}_1) + 2\alpha R_2 \frac{g_1(x)}{g_1(x) + \tilde{g}_1(x)}, \tag{34}$$

and the + sign in equations (32) and (33) is taken if the function $\mathcal{F}(x) > 0$. Similarly, the $-$ sign is chosen if $\mathcal{F}(x) < 0$.

In addition, the consistency condition (31) simultaneously imposes extra constraints among the R coefficients. There are two possible relations. The first relation is for the case $R_2 = -\tilde{R}_2 > 0$, where R_0 and $\tilde{R}_0$ introduced in equation (20) are given by

$$R_0^{(1)} = \frac{1}{4R_2} R_1^2, \qquad \tilde{R}_0^{(1)} = \frac{1}{4R_2}\left[(R_1 + \tilde{R}_1)(R_1 + \tilde{R}_1 + 2\alpha R_2) - \tilde{R}_1^2\right]. \tag{35}$$

The superscript notation $^{(1)}$ stands for the first case. The same notation applies to the other case, too. The second relation is for the case $R_2 = -\tilde{R}_2 < 0$. We get

$$R_0^{(2)} = -\frac{1}{4R_2}\left[(R_1 + \tilde{R}_1)(R_1 + \tilde{R}_1 + 2\alpha R_2) - R_1^2\right], \qquad \tilde{R}_0^{(2)} = -\frac{1}{4R_2} \tilde{R}_1^2. \tag{36}$$

In fact, the two relations are not really independent. Because when R_2 is changed to $-R_2 = \tilde{R}_2$, we observe that the roles of $g_1(x)$ and $\tilde{g}_1(x)$ in equation (21) get exchanged accordingly. This could further lead to the exchange of the both superpotentials given in equation (19) in a similar fashion. We therefore expect that there exists a symmetrical transformation between the two cases. As a result, we can map all the eigenvalue problems, including the eigenenergies, eigenfunctions, superpotentials and potentials, in the first case ($R_2 > 0$) to those in the second case ($\tilde{R}_2 = -R_2 < 0$). Explicitly, the transformation rule is described by the following changes:

$$\begin{aligned}
&g_1(x) \leftrightarrow \tilde{g}_1(x), \qquad g_0(x) \to \tilde{g}_0(x), \qquad \tilde{g}_0(x) \to g_0(x) + \alpha g_1(x),\\
&R_2 \leftrightarrow \tilde{R}_2, \qquad R_1 \to \tilde{R}_1, \qquad \tilde{R}_1 \to R_1 + 2\alpha R_2,\\
&R_0^{(1)} \to \tilde{R}_0^{(2)}, \qquad \tilde{R}_0^{(1)} \to R_0^{(2)} + \alpha R_1 + \alpha^2 R_2.
\end{aligned} \tag{37}$$

Under such a transformation, it can be easily checked that the set of equations, equations (21)–(25), remains intact.

The transformation rule (37) actually has an important implication on the Hilbert space of the system. To see this, let us consider a generic shape invariant potential in two steps. If $R_2 > 0$ is considered, we then find, for example, that the eigenfunctions of the potential $V_-(x, a_1)$ are ordered in such a way as those depicted in equations (17) and (18) and the energy eigenvalues are ordered as those represented in equations (15) and (16). As we shall show in subsection 3.3, this indeed is the case for the shape invariant potentials in two steps. To be more specific, the ground-state wavefunction of the system is $\psi_0^{(-)}(x, a_1) \propto \exp[-\int^x W(y, a_1)\,dy]$, the first excited state is $\psi_1^{(-)}(x, a_1) \propto A^\dagger(a_1)\exp[-\int^x \tilde{W}(y, a_1)\,dy]$, and so on. The superpotentials $W(x, a_1)$ and $\tilde{W}(x, a_1)$ are given in equation (19). Moreover, the first three remainders are given by $R^{(1)}(a_1) = R_0^{(1)} + R_1 a_1 + R_2 a_1^2$, $\tilde{R}^{(1)}(a_1) = \tilde{R}_0^{(1)} + \tilde{R}_1 a_1 + \tilde{R}_2 a_1^2$ and $R^{(1)}(a_2) = R_0^{(1)} + R_1 a_2 + R_2 a_2^2$, where $a_2 = a_1 + \alpha$ and equation (35) are used.

Having found all the results for the $R_2 > 0$ case, we can readily obtain, without doing any calculation, those for the $R_2 < 0$ case by simply applying the transformation rule (37) on the former case. Upon performing the transformation, we find that the superpotentials transform as $W(x, a_1) \to \tilde{W}(x, a_1)$ and $\tilde{W}(x, a_1) \to W(x, a_2)$ and that the remainders transform as $R^{(1)}(a_1) \to \tilde{R}^{(2)}(a_1)$, $\tilde{R}^{(1)}(a_1) \to R^{(2)}(a_2)$, $R^{(1)}(a_2) \to \tilde{R}^{(2)}(a_2)$, and so on. That is to say, according to the transformation rule the ground-state wavefunction at this time is transformed to $\tilde{\psi}_0^{(-)}(x, a_1) \propto \exp[-\int^x \tilde{W}(y, a_1)\,dy]$ and the first excited state is to $\psi_1^{(-)}(x, a_1) \propto \tilde{A}^\dagger(a_1)\exp[-\int^x W(y, a_2)\,dy]$. As a result, for the case $R_2 < 0$, the entire spectrum of eigenenergies of the transformed potential $\tilde{V}_-(x, a_1)$ is given by

$$E_{2n}^{(-)} = \sum_{k=1}^{n} [\tilde{R}^{(2)}(a_k) + R^{(2)}(a_{k+1})], \tag{38}$$

$$E_{2n+1}^{(-)} = \sum_{k=1}^{n} [\tilde{R}^{(2)}(a_k) + R^{(2)}(a_{k+1})] + \tilde{R}^{(2)}(a_{n+1}), \tag{39}$$

where $E_0^{(-)} = 0$, and the corresponding unnormalized eigenfunctions of the system are

$$\psi_{2n}^{(-)}(x, a_1) \propto [\tilde{A}^\dagger(a_1)A^\dagger(a_2)] \cdots [\tilde{A}^\dagger(a_n)A^\dagger(a_{n+1})]\tilde{\psi}_0^{(-)}(x, a_{n+1}), \tag{40}$$

$$\psi_{2n+1}^{(-)}(x, a_1) \propto [\tilde{A}^\dagger(a_1)A^\dagger(a_2)] \cdots [\tilde{A}^\dagger(a_n)A^\dagger(a_{n+1})]\tilde{A}^\dagger(a_{n+1})\psi_0^{(-)}(x, a_{n+2}). \tag{41}$$

The Hilbert space of the $R_2 < 0$ case can thus be completely determined by that of the $R_2 > 0$ case.

Let us now analyze the general properties of energy spectrum of the shape invariant potentials in two steps with $a_{n+1} = a_n + \alpha = a_1 + n\alpha$. Using the explicit forms of equation (20), we write the dependence of the remainders $R^{(i)}(a_{n+1})$ and $\tilde{R}^{(i)}(a_{n+1})$ on the quantum number n as

$$\begin{aligned} R^{(i)}(a_{n+1}) &\equiv R_0^{(i)} + R_1 a_{n+1} + R_2 a_{n+1}^2 \\ &= R^{(i)}(a_1) + \alpha(R_1 + 2a_1 R_2)n + \alpha^2 R_2 n^2, \end{aligned} \tag{42}$$

$$\begin{aligned} \tilde{R}^{(i)}(a_{n+1}) &\equiv \tilde{R}_0^{(i)} + \tilde{R}_1 a_{n+1} - R_2 a_{n+1}^2 \\ &= \tilde{R}^{(i)}(a_1) + \alpha(\tilde{R}_1 - 2a_1 R_2)n - \alpha^2 R_2 n^2, \end{aligned} \tag{43}$$

where i can be 1 or 2. We note that both remainders in equations (42) and (43) are quadratic functions of the quantum number n and the coefficients of the n^2-term are of the opposite sign. Hence, either $R^{(i)}(a_{n+1})$ or $\tilde{R}^{(i)}(a_{n+1})$ shall become negative for large enough quantum number n. However, negative values of the remainder are definitely not acceptable in the energy spectrum at all, because it results in the level crossing in energy. We thus conclude that, for a general shape invariant potentials in two steps with $R_2 \neq 0$ in the translation class, the number of bound states must be finite and the corresponding potential must be of finite depth. There is one exception, however. It is for the potentials where $\alpha = 0$, since at this time the terms containing n vanish in the both remainders. As we will see below, the solvable potentials for $\alpha = 0$ are the so-called singular harmonic oscillators, and thus contain infinite number of bound states.

Besides, the both remainders (42) and (43) depend differently on the quantum number n in a specific way. For the case $R_2 > 0$, we note by using equation (35) that the remainder $R^{(1)}(a_{n+1})$ is a concave-up parabolic function of n with a double root ($R^{(1)}(a_{n+1}) = 0$) appearing at

$$n_0^{(1)} = -\frac{R_1 + 2a_1 R_2}{2\alpha R_2}, \tag{44}$$

and the remainder $\tilde{R}^{(1)}(a_{\tilde{n}+1})$ is a concave-down parabolic function of n whose two roots ($\tilde{R}^{(1)}(a_{\tilde{n}+1}) = 0$) are located, separately, at

$$\tilde{n}_{\pm}^{(1)} = \frac{1}{2\alpha R_2}\left[(\tilde{R}_1 - 2a_1 R_2) \pm \frac{|\alpha|}{\alpha}\sqrt{(R_1 + \tilde{R}_1)(R_1 + \tilde{R}_1 + 2\alpha R_2)}\right]. \tag{45}$$

It is obvious that, for both roots $\tilde{n}_{\pm}^{(1)}$ to be real, we must require that the product $(R_1 + \tilde{R}_1)(R_1 + \tilde{R}_1 + 2\alpha R_2) \geqslant 0$.

In the same vein, for the other case $R_2 < 0$, we obtain by equation (36) that the remainder $\tilde{R}^{(2)}(a_{\tilde{n}+1})$ instead is a concave-up parabolic function of n with a double root ($\tilde{R}^{(2)}(a_{\tilde{n}+1}) = 0$) at

$$\tilde{n}_0^{(2)} = \frac{\tilde{R}_1 - 2a_1 R_2}{2\alpha R_2}, \tag{46}$$

and the remainder $R^{(2)}(a_{n+1})$ a concave-down parabolic function of n with two distinct roots ($R^{(2)}(a_{n+1}) = 0$) at

$$n_{\pm}^{(2)} = -\frac{1}{2\alpha R_2}\left[(R_1 + 2a_2 R_2) \pm \frac{|\alpha|}{\alpha}\sqrt{(R_1 + \tilde{R}_1)(R_1 + \tilde{R}_1 + 2\alpha R_2)}\right], \tag{47}$$

where $(R_1 + \tilde{R}_1)(R_1 + \tilde{R}_1 + 2\alpha R_2) \geqslant 0$ must be satisfied as before. It is noted that equations (46) and (47) can be directly gotten from equation (44) and (45) by applying the transformation rule (37), respectively.

We are now in a position to present the solutions of the two-step shape invariant superpotentials by solving the differential equations (27) or (28). Without loss of generality, we choose $R_2 = -\tilde{R}_2 > 0$, since the result for the $R_2 < 0$ case can be easily deduced by the transformation rule (37). In addition, we also choose $R_1 + \tilde{R}_1 > 0$ and $R_1 + \tilde{R}_1 + 2\alpha R_2 > 0$ in the solutions to be presented below. Comments concerning the choice $R_1 + \tilde{R}_1 < 0$ will be given whenever needed. It turns out when $R_2 \neq 0$ that two SUSY preserving solutions can be constructed. The first solution is for the case $\alpha = 0$, which is shown to be the singular harmonic oscillator potentials. The other solution is for the case $\alpha < 0$, which results in a new class of solvable potentials of shape invariance in two steps, that is not discussed previously. Before presenting the calculations, let us emphasize one important point in regard to the unbroken SUSY: only those superpotentials $W(x, a_1)$ and $\tilde{W}(x, a_1)$ are admissible that give rise to square integrable zero-energy ground-state wavefunctions $\psi_0^{(-)}(x, a_1)$ and $\tilde{\psi}_0^{(-)}(x, a_1)$. If it cannot be fulfilled, then SUSY is spontaneously broken.

3.1. The solution of $\alpha = 0$

The first SUSY retaining solution is for the case $\alpha = 0$, therefore we have $a_{n+1} = a_1$, for $n \geqslant 1$. The parameter is not shifted at all. Using equation (20), we obtain the remainders with the properties $R(a_{n+1}) = R(a_1)$ and $\tilde{R}(a_{n+1}) = \tilde{R}(a_1)$. From equations (15) and (16), the energy eigenvalues are given by $E_{2k}^{(-)} = k(R(a_1) + \tilde{R}(a_1))$ and $E_{2k+1}^{(-)} = E_{2k}^{(-)} + R(a_1)$ for $k = 0, 1, 2, \ldots$. Hence, the energy spectrum of the system consists of two shifted sets of equally spaced eigenvalues. This is nothing but the energy spectrum of the singular harmonic oscillator [13].

Explicitly, equations (21)–(25) are relatively easy to solve when $\alpha = 0$, so we will solve them directly. After performing the integrals and setting the integration constants zero, we get the g-functions:

$$\left.\begin{matrix} g_0(x) \\ \tilde{g}_0(x) \end{matrix}\right\} = \frac{1}{4}(R_0 + \tilde{R}_0)x \pm \frac{1}{2}\frac{R_0 - \tilde{R}_0}{(R_0 + \tilde{R}_0)x}, \tag{48}$$

$$\left.\begin{matrix} g_1(x) \\ \tilde{g}_1(x) \end{matrix}\right\} = \frac{1}{4}(R_1 + \tilde{R}_1)x \pm \frac{R_2}{(R_1 + \tilde{R}_1)x}, \tag{49}$$

and a relation among the R-coefficients

$$\frac{R_1 - \tilde{R}_1}{R_1 + \tilde{R}_1} - \frac{R_0 - \tilde{R}_0}{R_0 + \tilde{R}_0} = \frac{2R_2(R_0 + \tilde{R}_0)}{(R_1 + \tilde{R}_1)^2}. \tag{50}$$

When substituting these results into equation (19), we again find that the resultant superpotentials $W(x, a_1) = g_0(x) + g_1(x)a_1$ and $\tilde{W}(x, a_1) = \tilde{g}_0(x) + \tilde{g}_1(x)a_1$ are those of the singular harmonic oscillator.

In addition, both superpotentials constructed above are found continuous with well-defined derivatives everywhere in either $x > 0$ or $x < 0$ regions. However, at the origin $x = 0$, they are singular and have an infinite discontinuity. The singular term of $W(x, a_1)(\tilde{W}(x, a_1))$ around $x = 0$ point behaves like $+(-)\frac{1}{2x}\left(\frac{R_0 - \tilde{R}_0}{R_0 + \tilde{R}_0} + \frac{2a_1 R_2}{R_1 + \tilde{R}_1}\right)$, respectively. To maintain unbroken SUSY and shape invariance, we have to restrict the strength of the both singular terms to be in the domain $-1 < \frac{R_0 - \tilde{R}_0}{R_0 + \tilde{R}_0} + \frac{2a_1 R_2}{R_1 + \tilde{R}_1} < 1$ [13, 17]. As a result, the singularities become the so-called 'soft' and the corresponding potentials are said to be 'transitional', since the wavefunctions of same energy defined in both $x > 0$ and $x < 0$ halves can have a chance to properly communicate to each other. Besides the singularity, for any one of the superpotentials, the infinite discontinuity that will generate an ill-defined derivative at the $x = 0$ point is definitely not acceptable. A regularization that preserves SUSY and shape invariance is needed, and in fact has been proposed [13]. It is shown that, to regularize the infinite discontinuity of the superpotential $W(x, a_1)$ (a similar regularization also applies to the superpotential $\tilde{W}(x, a_1)$), we consider instead a regularized, continuous superpotential $W^{\text{reg}}(x, a_1, \epsilon)$ that reduces to the original $W(x, a_1)$ in the limit $\epsilon \to 0$ as

$$W^{\text{reg}}(x, a_1, \epsilon) = W(x, a_1) f(x, \epsilon), \tag{51}$$

where $f(x, \epsilon) = \tanh^2 \frac{x}{\epsilon}$ is a moderating factor that is unity everywhere except in a small region around $x = 0$. It is introduced to provide a smooth interpolation through the discontinuity. In the limit $\epsilon \to 0$, the corresponding potential $V_-^{\text{reg}}(x, a_1, \epsilon)$ (3) derived from the superpotential $W^{\text{reg}}(x, a_1, \epsilon)$ will reduce to this form [13]

$$V_-^{\text{reg}}(x, a_1) = V_-(x, a_1) - 4W(x, a_1)\frac{x}{|x|}\delta(x). \tag{52}$$

The upshot is that the regularized potential $V_-^{\text{reg}}(x, a_1)$ now exhibits an extra singularity with Dirac delta-function behavior at the origin over the unregularized one $V_-(x, a_1)$.

3.2. The solution of $\alpha > 0$

Potentially, the second solution to equation (27) (or equation (28)) is for the $\alpha > 0$ case. However, it cannot be a true solution of the shape invariance condition in two steps, since it contains infinite number of bound states and therefore violates our earlier assertion. To see this, we note on solving equation (27) that the $g_1(x)$ function is given by the transcendental function below (for $x > 0$)

$$\frac{|\alpha|}{k}x = \tan^{-1}[kg_1(x)] + \frac{1}{\sqrt{1+k^2R_2}}\tan^{-1}\left[k\sqrt{\frac{g_1(x)^2 - R_2}{1+k^2R_2}}\right], \tag{53}$$

where $k = \sqrt{\frac{2|\alpha|}{R_1+\tilde{R}_1}}$. A similar expression can be obtained for the function $\tilde{g}_1(x)$ by equation (28). Here, $g_1(x) > 0$ and $\tilde{g}_1(x) > 0$ is assumed. It is stressed that the other choice $g_1(x) > 0 > \tilde{g}_1(x)$ renders the both functions being asymmetrical about $x = 0$, and is discarded.

As a matter of fact, the $g_1(x)$ function given above is only defined in the range $(x_1 < x < x_2)$, with $x_1 = \frac{k}{|\alpha|}\tan^{-1}\left[kR_2\right]$ and $x_2 = \frac{\pi}{2}\frac{k}{|\alpha|}[1 + (1 + k^2R_2)^{-1/2}]$. To extend the definition of the $g_1(x)$ function to the small positive x space, that is, in the $(0 < x < x_1)$ range, technically we need another transcendental function, where the arctan functions in equation (53) have to be replaced by the minus of arccot ones. Moreover, to further extend the function $g_1(x)$ to cover the negative x region $(x < 0)$, then we can consult the antisymmetric property of superpotentials, if the $g_1(x)$ function for $x > 0$ has been known.

Nevertheless, we are not really interested in other portions of the complete $g_1(x)$ function that are not presented here, since the part of $g_1(x)$ given in equation (53) alone can be shown not to be a solution of the shape invariance condition in two steps. The reason is as follows. The function $g_1(x)$ in equation (53) is a monotonically increasing function in the range $(x_1 < x < x_2)$, with the respective end values $g_1(x_1) = \sqrt{R_2}$ and $g_1(x_2) = \infty$. The $g_1(x)$ function is of infinite depth! So must be the corresponding superpotentials as well as the potentials constructed from it. That is to say, the potentials in problem allow an infinite number of bound states. This is contradictory to our earlier claim on the general structure of energy spectrum of the two-step shape invariant potentials $(\alpha \neq 0)$, where only a finite number of bound states is allowed. We thus conclude that no solution can exist for the shape invariant potentials in two steps, if $\alpha > 0$.

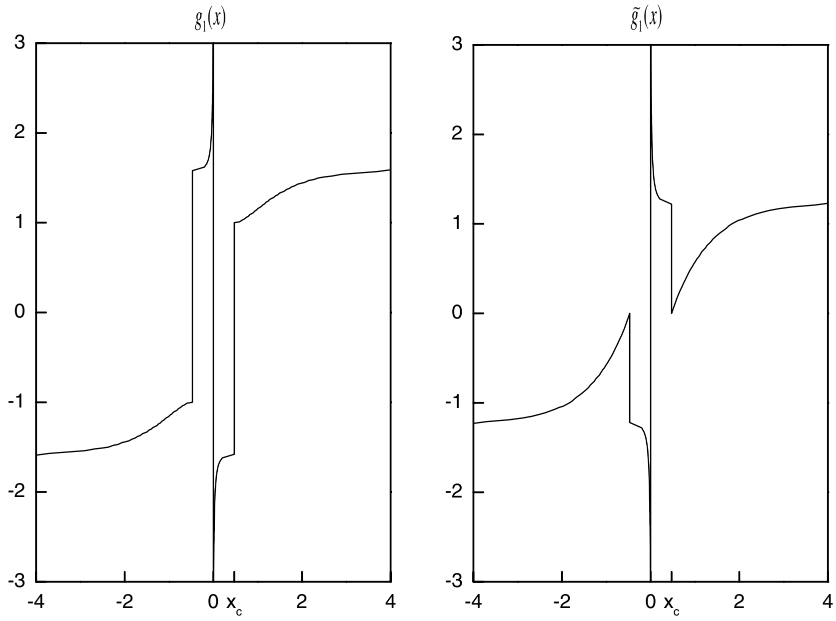

Figure 1. The functions $g_1(x)$ and $\tilde{g}_1(x)$ of shape invariance condition in two steps for the set of parameters $a_1 = 1, \alpha = -1, R_2 = 1, R_1 = 3$ and $\tilde{R}_1 = 2$. Here, $g_1(x)$ is given by equation (59), corresponding to equation (54) for $x > x_c$ and to equation (55) for $0 < x < x_c$, where $x_c \approx 0.471$. For $x < 0$, both functions are obtained by antisymmetrization.

3.3. The solution of $\alpha < 0$

The third solution of equation (27) (or equation (28)) to be discussed is for the $\alpha < 0$ case. This case will render the new class of solvable potentials of shape invariance condition in two steps, which retain SUSY. We shall present below the solution of $g_1(x)$ function only in the $x > 0$ half-axis. The solution in the other half-axis ($x < 0$) can be easily deduced from the first half by the antisymmetric property of superpotentials. It is found that the SUSY preserving $g_1(x)$ function (for $x > 0$), which is singular as $x \to 0$ and acquires a finite value as $x \to \infty$, can be constructed by patching two disjointed portions of solutions. At the position where $x = x_c > 0$ in between these two portions, there is a finite discontinuity. Consequently, as regard to the profile of the complete $g_1(x)$ function that covers the entire x axis, an infinite discontinuity is found occurring at $x = 0$, in addition to two finite discontinuities that are located separately at $x = \pm x_c$.

By means of equation (27), the first portion of the solution, denoted by $g_1^>(x)$, is defined in the range ($x_c < x < \infty$) and is given by the transcendental function below

$$\frac{|\alpha|}{k}x = \tanh^{-1}\left[k g_1^>(x)\right] + \frac{1}{\sqrt{1-k^2R_2}}\tanh^{-1}\left[k\sqrt{\frac{g_1^>(x)^2 - R_2}{1-k^2R_2}}\right], \tag{54}$$

where k is defined after equation (53). A similar expression can be deduced for the function $\tilde{g}_1^>(x)$ by solving equation (28). The choice of $g_1^>(x) > 0$ and $\tilde{g}_1^>(x) > 0$ is required for equation (54) to be an acceptable solution in the first portion. Here, x_c is the point such that $g_1^>(x_c) = \sqrt{R_2}$ or $x_c = \frac{k}{|\alpha|}\tanh^{-1}[kR_2]$. In figure 1, we plot both functions $g_1(x)$ and $\tilde{g}_1(x)$ in the entire x space. As we can see, $g_1(x)$ is a monotonically increasing function in the range

$(x_c < x < \infty)$ (so is the $\tilde{g}_1(x)$ function), with the respective end values $g_1^>(x_c) = \sqrt{R_2}$ and $g_1^>(\infty) = k^{-1}$. Since k^{-1} is finite and $R_1 + \tilde{R}_1 + 2\alpha R_2 > 0$ is also true, we conclude that the superpotentials $W(x, a_1)$ and $\tilde{W}(x, a_1)$ (19) constructed respectively from $g_1(x)$ and $\tilde{g}_1(x)$ are guaranteed to be of finite depth. Consequently, the corresponding potentials $V_-(x, a_1)$ and $\tilde{V}_-(x, a_1)$ will also be of finite depth and generate only a finite number of bound states. This is consistent with the desired structure of energy spectrum for the $\alpha \neq 0$ shape invariant potentials in two steps, which was mentioned previously.

Because the solution $g_1^>(x)$ (equation (54)) is given only in the $(x > x_c)$ range, it cannot be directly extended to the small x region $(x \to 0^+)$. For a complete function $g_1(x)$ to cover the entire space, we need a second portion of the solution for $g_1(x)$. To accomplish this, let us denote the second portion of the solution by $g_1^<(x)$. Then, on solving equation (27), we obtain the $g_1^<(x)$ function in the range $(0 < x < x_c)$ as

$$\frac{|\alpha|}{k}x = \coth^{-1}\left[k g_1^<(x)\right] + \frac{1}{\sqrt{1-k^2R_2}}\coth^{-1}\left[k\sqrt{\frac{g_1^<(x)^2 - R_2}{1-k^2R_2}}\right], \tag{55}$$

where the choice of $g_1^<(x) < 0$ and $\tilde{g}_1^<(x) > 0$ is required for this second portion of solution to retain unbroken SUSY.[3] Again, a similar expression can be gotten for the function $\tilde{g}_1^<(x)$. $g_1^<(x)$ and $\tilde{g}_1^<(x)$ are plotted in figure 1, respectively represented by the $g_1(x)$ and $\tilde{g}_1(x)$ curves over the specified range $(0 < x < x_c)$.

Here, the choice of $g_1^<(x) < 0$ and $\tilde{g}_1^<(x) > 0$ is very critical for the construction of normalizable ground-state wavefunctions $\psi_0^{(-)}(x, a_1)$ and $\tilde{\psi}_0^{(-)}(x, a_1)$ near the $x = 0$ point. To confirm this point, let us find a small x expression for equation (55), which is equivalent to expand both arccoth functions on the right-hand side of equation (55) at large value of $g_1^<(x)$. At this stage, we should have a formal series expansion for the position function as $x = x\left(\frac{1}{g_1^<}\right)$. Then, we can directly invert this function term by term to yield a series expansion for $g_1^<(x)$. After the smoke clears, we obtain a small x expansion for the $g_1^<(x)$ function of the form

$$g_1^<(x) = -\left(\frac{R_2}{6|\alpha|x}\right)^{1/3}\left[1 + \frac{4+3k^2R_2}{20k^2}\left(\frac{6|\alpha|x}{R_2}\right)^{2/3} + \mathcal{O}(x^{4/3})\right]. \tag{57}$$

Note that for small x the $g_1^<(x)$ function exhibits a $-x^{-1/3}$ singularity. In the same manner, a small x expansion for the $\tilde{g}_1^<(x)$ function can also be derived. It turns out that its series expansion is

$$\tilde{g}_1^<(x) = \left(\frac{R_2}{6|\alpha|x}\right)^{1/3}\left[1 + \frac{4-3h^2R_2}{20h^2}\left(\frac{6|\alpha|x}{R_2}\right)^{2/3} + \mathcal{O}(x^{4/3})\right], \tag{58}$$

where $h \equiv \sqrt{\frac{2|\alpha|}{R_1+\tilde{R}_1+2\alpha R_2}}$. Let us note that the $\tilde{g}_1^<(x)$ function is of the $x^{-1/3}$ singularity for small x, while having the same strength of singularity as the $g_1^<(x)$ function.

Hence, to examine the singular behaviors near the origin for the both superpotentials $W(x, a_1)$ and $\tilde{W}(x, a_1)$, we can simply substitute both series expansions of equations (57)

[3] We could have instead chosen $g_1^<(x) > 0$ and $\tilde{g}_1^<(x) > 0$ in the second portion in equation (55), since it is also a legitimate solution to equation (27). However, it would break SUSY spontaneously. Such a choice gives the superpotentials $W(x, a_1)$ and $\tilde{W}(x, a_1)$ in equation (19) singularities around $x = 0$ point of the forms, respectively,

$$\left(\frac{1}{2} - \frac{R_1 + 2a_1R_2}{\alpha R_2}\right)\frac{1}{x} \qquad \text{and} \qquad -\left(\frac{1}{2} + \frac{R_1 + 2a_1R_2}{\alpha R_2}\right)\frac{1}{x}. \tag{56}$$

Any arbitrary combination of $R_1 + 2a_1R_2 \neq 0$ thus results in both $\psi_0^{(-)}(x, a_1)$ and $\tilde{\psi}_0^{(-)}(x, a_1)$ not normalizable, and breaks SUSY as told. In order for these two superpotentials to retain unbroken SUSY, we must require $R_1+2a_1R_2 = 0$. However, by using equations (20) and (35) this implies that $R(a_1) = 0$. It leads us to a trivial SUSY shape invariant potential, where except for the zero-energy ground state no other excited state exists.

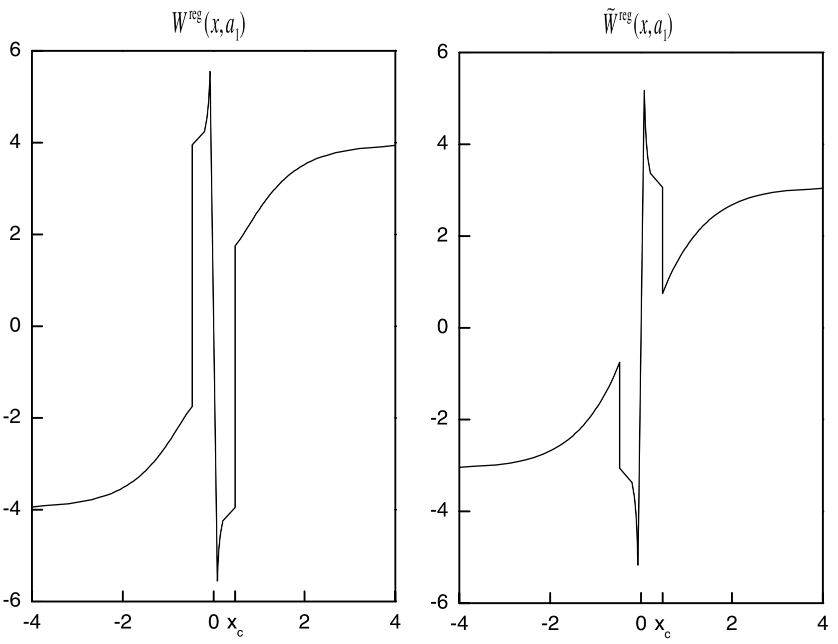

Figure 2. The corresponding regularized superpotentials $W^{\rm reg}(x, a_1)$ and $\tilde{W}^{\rm reg}(x, a_1)$, given by equations (19), (32) and (33), for the same set of parameters as in figure 1.

and (58), together with equations (32) and (33), into equation (19). After some algebra, we find that the singular term of $W(x, a_1)(\tilde{W}(x, a_1))$ around $x = 0$ point behaves like $-(+)\left[\frac{1}{24R_2x} + \mathcal{O}(x^{-1/3})\right]$, respectively. In order for the both ground-state wavefunctions $\psi_0^{(-)}(x, a_1)$ and $\tilde{\psi}_0^{(-)}(x, a_1)$ to be normalizable and for SUSY to still be retained, we must therefore restrict the strength of the singularities to be in the region, $0 < \frac{1}{12R_2} < 1$ [13, 17]. Note that $R_2 > 0$ here. Consequently, as long as $R_2 > \frac{1}{12}$ in equation (55) is satisfied, we are definitely guaranteed to obtain the new solution of SUSY preserving shape invariant potentials in two steps. Amongst, the main ingredient function $g_1(x)$ is composed of $g_1^{>}(x)$ (54) for $x_c < x < \infty$ and $g_1^{<}(x)$ (55) for $0 < x < x_c$. In mathematical expression, it is of the form (for $x > 0$)

$$g_1(x) = g_1^{<}(x)[1 - \theta(x - x_c)] + g_1^{>}(x)\theta(x - x_c), \tag{59}$$

where $\theta(x - x_c)$ is the step function, which is 0 for $x < x_c$ and is +1 for $x > x_c$. We note that, in the new class of shape invariant potentials in two steps, the function $g_1(x)$ $(\tilde{g}_1(x))$ cannot be expressed in terms of elementary functions, but only given in implicit forms (54) and (55). The same must be true for the corresponding superpotentials and potentials constructed out of it.

In figure 2, we show the respective regularized superpotentials $W^{\rm reg}(x, a_1)$ and $\tilde{W}^{\rm reg}(x, a_1)$ constructed from $g_1(x)$ and $\tilde{g}_1(x)$ numerically. Without regularization, both of them are found to exhibit an infinite discontinuity at $x = 0$, in addition to two finite ones at $x = \pm x_c$. Since the infinite discontinuity of the superpotentials is not acceptable, we thus need to regularize them by the method depicted in subsection 3.1 [13]. In figures 3 and 4, we plot the corresponding regularized potentials $V_-^{\rm reg}(x, a_1)$ and $\tilde{V}_-^{\rm reg}(x, a_1)$ and their energy spectra, respectively. For each regularized potential, three Dirac delta-function singularities are found: one is from the

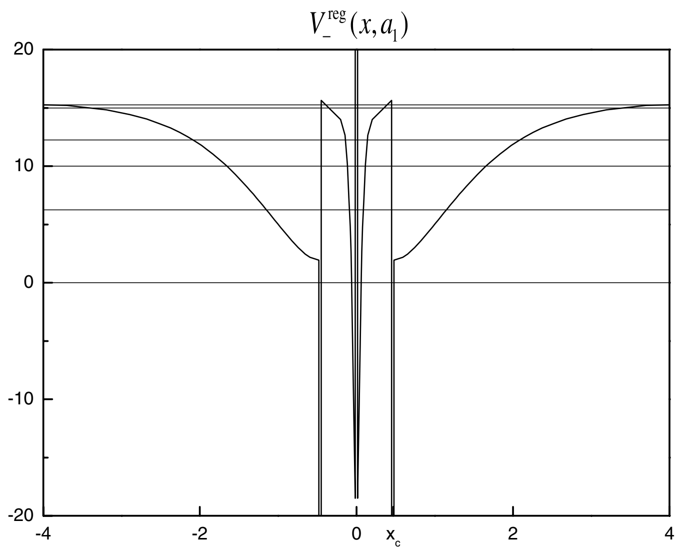

Figure 3. The corresponding regularized potential $V_-^{\mathrm{reg}}(x, a_1)$ and its eigenvalue spectrum, having six bound-state energies $E_n = 0$, 6.25, 10, 12.25, 15, 15.25 that are obtained by equations (15), (16), (42) and (43).

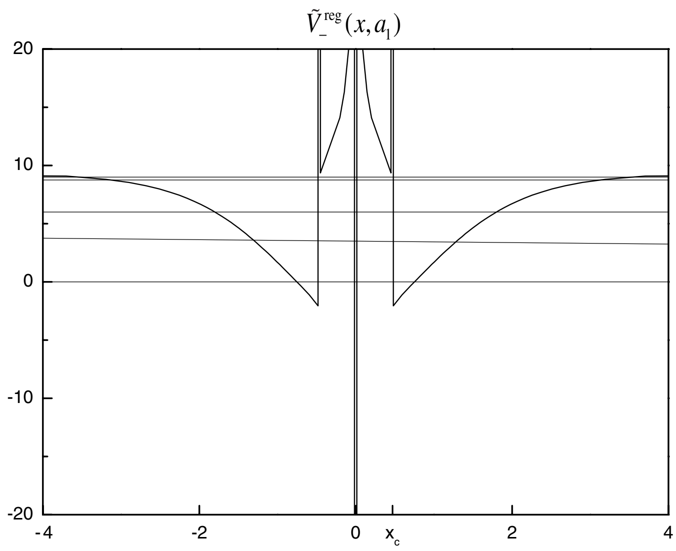

Figure 4. The corresponding regularized potential $\tilde{V}_-^{\mathrm{reg}}(x, a_1)$ and its eigenvalue spectrum, with five bound-state energies $E_n = 0$, 3.75, 6, 8.75, 9.

regularization of the infinite discontinuity at $x = 0$. The other two are from the derivatives of the finite discontinuities, expressed by the θ functions in equation (59), at $x = \pm x_c$. Note that the delta-functions are present to accommodate the zero-energy ground state for the regularized potentials.

Let us end the section with a discussion on the energy spectrum of the new class of shape invariant potentials in two steps. For unbroken SUSY and the choices $R_2 > 0$ and $R_1 + \tilde{R}_1 > 0$, the calculations indeed show that the depth of $V_-^{\mathrm{reg}}(x, a_1)$ is deeper than that of $\tilde{V}_-^{\mathrm{reg}}(x, a_1)$. See figures 3 and 4 for details. In this circumstance, the eigenfunctions of the system are ordered as those depicted in equations (17) and (18) and the energy eigenvalues are as those represented in equations (15) and (16).[4] Hence, according to the transformation rule (37), we conclude that for the case of $R_2 < 0$ and $R_1 + \tilde{R}_1 > 0$ the eigenfunctions of the system must be given in equations (40) and (41) and the energy eigenvalues must be represented in equations (38) and (39), instead. The very same results we have argued previously in the two paragraphs after equation (37).

Because the two-step shape invariant potentials are of finite depth ($R_2 > 0$) in the new class, the highest-energy bound state could be an even number state $\psi_{2N}^{(-)}(x, a_1)$ or an odd number state $\psi_{2N+1}^{(-)}(x, a_1)$, for N a positive integer. We have to determine under what condition the highest-energy eigenstate is even or odd. To proceed, we recall some important properties on the remainders implied from the paragraph in equations (44) and (45). The remainder $R^{(1)}(a_{n+1})$ is a concave-up function with $R^{(1)}(a_{n+1}) > 0$ for $n \neq n_0^{(1)}$ and $R^{(1)}(a_{n+1}) = 0$ for $n = n_0^{(1)}$. The remainder $\tilde{R}^{(1)}(a_{n+1}) > 0$ is a concave-down function with $\tilde{R}^{(1)}(a_{\tilde{n}+1}) > 0$ only in the interval $\tilde{n}_-^{(1)} \leqslant \tilde{n} \leqslant \tilde{n}_+^{(1)}$, where $\tilde{n}_-^{(1)} < 0$ and $\tilde{n}_+^{(1)} > 0$ must be satisfied to prevent negative values of the remainders at small quantum number ($n = 0, 1, \ldots$). As we already know, the quantum number definitely is a non-negative integer, but the roots $n_0^{(1)}$, $\tilde{n}_+^{(1)}$ and $\tilde{n}_-^{(1)}$ obtained from equations (44) and (45) may be integers or real numbers. So let us introduce a notation $[m]$ for them ($m = n_0^{(1)}, \tilde{n}_+^{(1)}$, and $\tilde{n}_-^{(1)}$) to denote the operation of taking the nearest integer that is equal to or larger than m. For instance, if $m = 5$ we have $[5] = 5$ and if $m = 4.32$ we have $[4.32] = 5$.

Now, we present the condition for the $R_2 > 0$ case on the even and odd highest-energy eigenstate as follows. A similar result can be reached for the $R_2 < 0$ case by the transformation rule (37). On the one hand, the system is found to exhibit an odd-number highest-energy eigenstate, say $\psi_{2N+1}^{(-)}(x, a_1)$. Then from the structure of energy spectrum (15) and (16), we impose the constraints on the remainders: $R^{(1)}(a_{n+1}) \neq 0$ (for $n = 0, 1, \ldots, N$) and $\tilde{R}^{(1)}(a_N) > 0$ and $\tilde{R}^{(1)}(a_{N+1}) \leqslant 0$. It consequently implies that $n_0^{(1)} \neq 0, 1, \ldots, N$ and the quantum number is defined by $N = [\tilde{n}_+^{(1)}]$. On the other hand, if the system exhibits an even-number highest-energy eigenstate, say $\psi_{2N}^{(-)}(x, a_1)$, then from equations (15) and (16), we get the constraints on the remainders as $R^{(1)}(a_{N+1}) = 0$ and $\tilde{R}^{(1)}(a_N) > 0$. This then implies that $\left[n_0^{(1)}\right] = n_0^{(1)}$ is an integer, the quantum number is given by $N = n_0^{(1)}$, and $[\tilde{n}_+^{(1)}] \geqslant N$. It is instructive to calculate these root values. Let us take the set of parameters in figure 1 as an example, where $a_1 = 1, \alpha = -1, R_2 = 1, R_1 = 3$ and $\tilde{R}_1 = 2$. By plugging them into equations (44) and (45), we get $n_0^{(1)} = 2.5$ and $\tilde{n}_\pm^{(1)} = \pm 1.936$. Because $n_0^{(1)}$ is not an integer, the quantum number becomes $N \equiv [\tilde{n}_+^{(1)}] = 2$. We therefore conclude that the highest-energy eigenstate of the system will be the odd number state $\psi_{2N+1}^{(-)}(x, a_1) = \psi_5^{(-)}(x, a_1)$. The result is in agreement with that in figure 3.

4. Conclusions

Using an ansatz for shape invariance condition in two steps, we have obtained by solving equations (21)–(25) a class of the SUSY preserving potentials, where the parameters a_1 and a_2 are related to each other by translation ($a_2 = a_1 + \alpha$). Due to the choice of the ansatz, the shape invariant potential presented in subsection 3.1 for the $\alpha < 0$ case looks cumbersome and complicated. However, there is a special limit for the ansatz that could render more

[4] Alternatively, if $R_1 + \tilde{R}_1 < 0$ is chosen, then it is the ground-state wavefunctions $\psi_0^{(+)}(x, a_1)$ and $\tilde{\psi}_0^{(+)}(x, a_1)$ that are normalizable.

nice-looking potentials. It is achieved by setting $R_2 = 0$ in equations (21)–(25). In this limit, we have either $\tilde{g}_1(x) = \pm g_1(x)$ from equation (21). The choice of $\tilde{g}_1(x) = -g_1(x)$ gives us trivial solution, since all the functions $g_i(x)$ and $\tilde{g}_i(x)$ $(i = 0, 1)$ are nothing but constants. We are thus lead to choose the relation $\tilde{g}_1(x) = g_1(x)$. By using this fact and simultaneously solving equations (22)–(25), we find that the functions $g_0(x)$ and $\tilde{g}_0(x)$ are expressible in terms of $g_1(x)$ by

$$\left.\begin{matrix} g_0(x) \\ \tilde{g}_0(x) \end{matrix}\right\} = \left(\frac{R_0}{R_1} \mp \frac{\alpha}{4}\right) g_1(x) \pm \frac{R_1 - \tilde{R}_1}{8 g_1(x)}. \tag{60}$$

In addition, the relation among the R coefficients is written as $\alpha R_1 \tilde{R}_1 = 2(R_1 \tilde{R}_0 - R_0 \tilde{R}_1)$. Meanwhile, the $g_1(x)$ function can be solved directly from equation (27) by setting $R_2 = 0$ and taking the + sign on the left-hand side. Then, based on the obtained result for the $g_1(x)$ function, we are able to construct the superpotentials $W(x, a_1)$ and $\tilde{W}(x, a_1)$ from equation (19) and derive the corresponding two-step shape invariant potentials.

It turns out that there are three solutions of interest that retain SUSY [13]. As an illustration, we present these solutions for $R_1 + \tilde{R}_1 > 0$, because it guarantees that the wavefunctions $\psi_0^{(-)}(x, a_1)$ and $\tilde{\psi}_0^{(-)}(x, a_1)$ can be made normalized. The first solution is when $\alpha = 0$, where we surprisingly discover the so-called singular harmonic oscillator superpotentials. The second solution is when $\alpha > 0$, where the singular Pöschl–Teller I superpotentials is reproduced. The last solution is when $\alpha < 0$, where this case yields us the singular Pöschl–Teller II superpotentials. In all three special solutions, the superpotentials are singular like $\pm \frac{1}{2x} \frac{R_1 - \tilde{R}_1}{R_1 + \tilde{R}_1}$ and have an infinite discontinuity at the origin. To retain unbroken SUSY and shape invariance, we have to restrict the strength of singularity and to regularize the infinite discontinuity. As a result, the regularized potentials acquire an extra Dirac delta-function singularity at the origin.

In the new translation class of the shape invariant potentials in two steps, the superpotentials $W(x, a_1)$ and $\tilde{W}(x, a_1)$ are found not to be expressible in terms of elementary functions, but only in implicit forms via the main ingredient function $g_1(x)$ (59). Meanwhile, the function $g_1(x)$ consists of two portions of transcendental functions $g_1^>(x)$ (54) and $g_1^<(x)$ (55). Despite this, both the superpotentials can be shown to exhibit an x^{-1} singularity with infinite discontinuity at the origin, in addition to two antisymmetric finite discontinuities at the $x = \pm x_c$ points. After proper regularization, the regularized potentials $V_-^{\rm reg}(x, a_1)$ and $\tilde{V}_-^{\rm reg}(x, a_1)$ thus acquire three Dirac delta-function singularities separately located at the $x = 0$ and $x = \pm x_c$ points. As a matter of fact, the construction of such superpotentials is nontrivial, since it involves patching different portions of the $g_1(x)$ and $\tilde{g}_1(x)$ functions to complete the work. In order for both superpotentials to retain SUSY and shape invariance, we must carefully adjust the relative sign between the $g_1(x)$ and $\tilde{g}_1(x)$ functions at small x, so that the singularities at the origin are still within the so-called 'soft' region and the potentials become 'transitional'. Furthermore, from the structure of energy spectrum, we conclude that the new shape invariant potentials in two steps allow only a finite number of bound states, so that they must be of finite depth. Therefore, any potential of infinite depth certainly cannot be a candidate of the shape invariant potential in two steps in this new translation class.

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